Weak measurement of the Dirac distribution

Jeff S. Lundeen[1,*] and Charles Bamber[1]

[1] *Institute for National Measurement Standards, National Research Council, 1200 Montreal Rd., Ottawa, ON, K1A 0R6*

Recent work [J.S. Lundeen et al. Nature, **474**, 188 (2011)] directly measured the wavefunction by weakly measuring a variable followed by a normal (i.e. 'strong') measurement of the complementary variable. We generalize this method to mixed states by considering the weak measurement of the product of the two observables, which, as a non-Hermitian operator, is normally unobservable. This generalized method provides mixed states an operational definition related to the operator representation proposed by Dirac. Uniquely, it can be performed 'in situ', determing the quantum state without destroying it.

PACS numbers: 03.65.Ta, 03.65.Wj, 42.50.Dv, 03.67.-a

The wavefunction Ψ is at the heart of quantum mechanics, yet its nature has been debated since its inception. It is typically relegated to being a calculational device for predicting measurement outcomes. Recently, Lundeen et al. proposed [1, 2] a simple and general operational definition of the wavefunction based on a method for its direct measurement: "it is the average result of a weak measurement of a variable followed by a strong measurement of the complementary variable." The 'wavefunction' referred to here was introduced along with the Schrdinger Equation and can be thought of as a special case of a quantum state, known as a 'pure state.' The general case is represented by the density operator ρ, which can describe both pure and 'mixed' states. The latter incorporates both the effects of classical randomness (e.g., noise) and entanglement with other systems (e.g., decoherence). The density operator plays an important role quantum statistics, quantum information, and the study of decoherence. As quantified by the Purity $\mu = \mathrm{Tr}[\rho^2] \leq 1$, a density operator can range from pure ($\mu = 1$) to completely mixed ($\mu = 1/N$, where N is the Hilbert space dimension). We investigate what becomes of our operational definition of the wavefunction when it is applied to general quantum states.

The standard method for experimentally determining the density operator is Quantum State Tomography [3]. In it, one makes a diverse set of measurements on an ensemble of identical systems and then determines the quantum state that is most compatible with the measurement results. An alternative is our direct measurement method, which may have advantages over tomography, such as simplicity, versatility, and directness. As compared to tomography, which works with mixed states, a significant limitation of the direct measurement method is that it has only been shown to work with pure states. The most common goal of state determination is to evaluate a system's difference from a target quantum state, which might be, for example, a potential resource for a quantum information protocol. Since the target is usually a pure state, this difference often due to mixedness, and so it can be quantified by the purity of the state.

In this paper, we consider whether our direct method can be generalized to density operators. Previous works have developed direct methods to measure quasi-probability distributions such as the Wigner function [4], Husimi Q-function [5], and the Glauber-Sudarshan P-function [6]. These are position-momentum (i.e. 'phase-space') distributions that are equivalent to the density operator, and have many, but not all, of the properties of a standard probability distribution. The most used quasi-probability distribution is the Wigner function $W_\rho(x,p)$, which has the following properties: (1) it is real; (2) less than one, $|W_\rho| \leq 1$; (3) its marginals give correct predictions for position x and momentum p probability distributions (i.e. $\int W_\rho(x,p)dx = \mathrm{Prob}(p)$); and (4) the mean value of an observable A is just an overlap, $\langle A \rangle = 2\pi\hbar \int\int W_\rho(x,p) W_A(x,p) dxdp$. However, unlike a standard probability W_ρ can be negative. This is what enables it to be compatible with the predictions of quantum mechanics. The Wigner function can be directly measured [7] by displacing the system in phase space and then measuring the parity operator (this is a nontrivial requirement, see [8]). Equivalently, the integral of the interference between a pair of rotated and displaced replicas of the system will give the Wigner function [9]. The Husimi Q-function can be directly measured by an eight-port homodyne apparatus or by projection on the harmonic oscillator ground state [10]. These phase-space distributions are created to be the closest quantum analogs to a classical probability distribution. In this sense, they are inherently amenable to direct measurement.

A lesser known phase-space distribution that can be used to represent a quantum operator was introduced by Dirac in 1945 in his paper, "On the Analogy Between Classical and Quantum Mechanics." The Dirac distribution has since been extended to discrete Hilbert spaces [11]. In various guises it has been investigated periodically during last half-century [12]. In optics, variations of the Dirac distribution have been used widely, appearing in Walther's definition of the radiance function in radiometry [13] and Wolf's specific intensity [14] (as pointed out in Ref. [11]). Dirac showed that an operator A could be represented by its overlap with the basis states of two non-commuting variables. Specifically, one can represent

an operator in phase-space as $S_A(x,p) = \langle x| A |p\rangle \cdot \langle p|x\rangle$. If $A = \rho$, the Dirac distribution is representation of the quantum state of a system and shares features (2) to (4), from above, with the Wigner function [11]. Unlike the Wigner function, it is compatible with Bayes' law and, thus, is consistent with a quantum analog of classical determinism [15]. Hence, it has many of the desired features of quasi-probability distributions but also has a novel logical consistency.

However, the Dirac distribution has two peculiar features: one, it is sensitive to the ordering of the non-commuting variables (i.e. $\langle x| A |p\rangle \cdot \langle p|x\rangle \neq \langle p| A |x\rangle \cdot \langle x|p\rangle$); and two, Dirac noted that although his distribution "was developed to provide a formal probability" for x and p "it turns out to be in general a complex number." Surprisingly, Dirac saw the sensitivity of his distribution to the arbitrary choice of ordering not as a drawback but, rather, as a desirable characteristic, since it emphasized the key difference between quantum and classical mechanics: non-commutivity of operators [16]. Even accepting this, the Dirac distribution's complex nature has likely inhibited its acceptance as a quasi-probability distribution. Other complex phase-space operator representations have proven useful in quantum optics [17]. And ways in which complex probabilities can be made logically consistent with a frequentist interpretation of probability have been outlined [18]. Surprisingly, despite being complex we will show that the Dirac distribution is directly measurable, much like the Wigner function. Notably, in our method, the choice of variable ordering has a clear physical meaning. Thus, it removes a degree of arbitrariness from the distribution's definition.

We begin by considering what happens to our method for directly measuring the wavefunction when the state is not pure. At the heart the direct method is weak measurement, which we now review. Upon a standard (i.e.'strong') measurement of a quantum system, the density operator changes depending on the measurement outcome. This change can be viewed in two ways: as a straightforward update of our knowledge of the state of the system or, at least in some situations [18], as a physical disturbance due to the interaction between the system and the measurement apparatus (e.g. the Heisenberg Microscope [19]). To perform a weak measurement one reduces the strength of the interaction between the measured system and the measurement apparatus. This results in a commensurate reduction in the amount of information one receives from the measurement and, also, a reduction in the disturbance induced by the measurement. This tradeoff is inherent in quantum mechanics: measurement precision for system disturbance [20]. While a weak measurement on a single system provides little information, by repeating it on an arbitrarily large ensemble of identical systems one can precisely determine the *average* measurement result with arbitrary precision. This average is simply the standard quantum expectation value, $\langle \Psi| A |\Psi\rangle$, where A is the observable measured and $|\Psi\rangle$ is the pure state of the system. This is true independent of the strength of the measurement.

Over the last decade, interest in weak measurement has grown as researchers have realized its potential for interrogating quantum systems in a coherent manner [21]. Weak measurement theory has been used to model and understand photonic phenomena in birefrigent photonic crystals [22], fiber networks [23], cavity QED [24], and quantum tunelling [25]. Weak measurement provides insight into a number of fundamental quantum effects, including the role of the uncertainty principle in the double-slit experiment [18, 26], the Legget-Garg inequality [27], the quantum box problem [28], and Hardy's paradox [29]. Weak measurement has also been used to amplify small experimental effects [30] and as feedback for control of a quantum system [31]. Weak measurements have been demonstrated in both classical [32] and non-classical systems [33].

A distinguishing feature of weak measurement is that, in the limit of zero interaction, the quantum state of the system remains unchanged. Subsequent measurements can now provide additional information about that initial quantum state. Consider a subsequent strong measurement of observable C that results in outcome c (corresponding to eigenstate eigenstate $|c\rangle$). The average result of the weak measurement of A in the sub-ensemble of systems giving $C = c$ is called the 'Weak Value' and is given by [34],

$$\langle A\rangle_{\Psi}^{c} = \frac{\langle c| A |\Psi\rangle}{\langle c|\Psi\rangle}. \tag{1}$$

Surprisingly, the weak value can be outside the range of the eigenvalues of A and can even be complex [35–37]. The concept of weak measurement is universal [38–40] but insight into the meaning of complex weak values can be gained by considering a particular model of measurement by Von Neumann [41]. In this general model, the measurement apparatus has a pointer that shifts in position to indicate the result of a strong measurement of A. In weak measurement, not only does the average position of pointer shift but also the average momentum of the pointer. These shifts are proportional to the real and imaginary parts of the weak value, respectively, thus giving them straightforward physical manifestations [35–37]. The complex nature of the weak value is what enables us to directly measure real and imaginary parts of the wavefunction and, as we show later, directly measure the Dirac distribution.

We now review our method for the direct measurement of the wavefunction. The concept is general, however here we consider the case of a discrete Hilbert space. In this space, one is free to choose the basis $\{|a\rangle\}$ (associated with observable A) in which the wavefunction will be measured. The method consists of weakly measuring a projector in this basis $\pi_a \equiv |a\rangle\langle a|$, and post-selecting on

a particular value b_0 of the complementary observable B. By 'complementary' we mean that $\langle a|b_0\rangle = 1/\sqrt{N}$ for all a, where N is the dimension of the Hilbert space. That is, the overlap is real and constant as function of a. The existence of state $|b_0\rangle$ is guaranteed by the existence of at least two mutually unbiased bases (MUB) in any Hilbert space [42]. Using Eq. (1), the quantum state $|\Psi\rangle$ is given by

$$|\Psi\rangle = v \cdot \sum_a \langle \pi_a \rangle_\Psi^{b_0} |a\rangle , \tag{2}$$

where $\langle \pi_a \rangle_\Psi^{b_0}$ is the weak value and v is a constant that is independent of a. Thus by stepping through the values of a in a series of weak measurements one can directly measure $|\Psi\rangle$ represented in the a basis.

To extend our method to mixed states we need to calculate the weak value of a system described by a density operator. There always exists an orthonormal basis $\{|\lambda\rangle\}$ in which,

$$\rho = \sum_\lambda p_\lambda |\lambda\rangle \langle\lambda| . \tag{3}$$

Thus, a mixed state can always be thought of the random preparation of a set of orthogonal pure states $|\lambda\rangle$, each with probability p_λ. Refs. [43] showed that the weak value will simply be a weighted sum of the weak values for every pure state in the decomposition given in Eq. (3):

$$\langle A \rangle_\rho^c = \sum_\lambda P(\lambda|c) \langle A \rangle_\lambda^c , \tag{4}$$

where $P(\lambda|c)$ is the probability of the initial state being prepared in $|\lambda\rangle$ given the system was post-selected in $|c\rangle$. Using Bayes' Theorem to simplify this (as in [43]) we arrive at:

$$\langle A \rangle_\rho^c = \frac{\langle c| A\rho |c\rangle}{\langle c| \rho |c\rangle}. \tag{5}$$

And applying this to our direct measurement method we find,

$$\langle \pi_a \rangle_\rho^{b_0} = \frac{\langle b_0|a\rangle \langle a| \rho |b_0\rangle}{\langle b_0| \rho |b_0\rangle}. \tag{6}$$

Examination of Eq. (6) shows that only $2N$ real parameters are found by scanning a. This will not generally be sufficient to determine all the parameters in ρ, which has $N^2 - 1$ real parameters. As might be suspected, our method for the direct measurement of the wavefunction cannot be used to determine a mixed state.

We now consider whether the direct measurement technique can be used for a more modest task: to determine whether a state is mixed. This could be used as an indicator for when the method should not be used. Or, if it is also a measure of state purity μ, it could be used to determine the quality of prepared quantum states. The weak value in Eq. (6) is equal to sum of one row of the density matrix, $\rho_{a_1 a_2} = \langle a_1 | \rho | a_2 \rangle$, i.e. $\langle \pi_{a_1} \rangle_\rho^{b_0} = \frac{1}{N \cdot \rho_{b_0 b_0}} \sum_{a_2}^N \rho_{a_1 a_2}$, where $\rho_{b_0 b_0}$ is the probability that the measurement of $B = b_0$. In effect, one measures a distinct weak value for each pure state in the statistical decomposition in Eq. (3). This simply adds noise to our determination of the weak value for the density operator. In the limit of zero interaction strength, the weak measurement exhibits maximal statistical uncertainty and so additional noise would be imperceptible. Thus, we conclude that the original direct measurement method does not contain any signature of the purity of the state.

We now consider what happens if one replaces the strong measurement of B with a weak measurement. Specifically, we investigate the weak measurement of the product of projectors from the two MUB, $S_{ab} \equiv |b\rangle\langle b|a\rangle\langle a|$. Since there is no longer any post-selection, this average is calculated in the same manner as the standard quantum expectation value:

$$\langle S_{ab} \rangle_\rho = \mathrm{Tr}\left[S_{ab}\rho\right] \tag{7}$$

$$= \langle a| \rho |b\rangle \langle b|a\rangle \tag{8}$$

$$= S_\rho(a,b), \tag{9}$$

where $S_\rho(a,b)$ is the discrete Hilbert space version of the Dirac distribution as defined in [11] (there exist analogous discrete Wigner functions [44]). This is the main result of the paper; A joint weak measurement of complementary variables A and B directly measures the Dirac distribution at values a and b. By scanning a and b, so as to measure the Dirac distribution for over all values of (a,b), one completely determines the density operator. Similarly, a joint weak measurement of $S_{xp} \equiv |p\rangle\langle p|x\rangle\langle x|$ on a mixed state ρ gives the phase-space version of the Dirac distribution, $S_\rho(a,b)$.

In order to calculate the density operator from the Dirac distribution one must know $\langle b|a\rangle = \exp(i\theta_{ab})/\sqrt{N}$. Unfortunately, other than in a select few cases, it is not generally known what are the bases in the MUB set (for any given Hilbert space). Thus, a general formula for θ_{ab} is also unknown. However, if $\{|a\rangle\}$ is taken to be the standard basis (i.e. $\sum_{a=0}^N |a\rangle\langle a| = I$, the identity operator) then one MUB, which we take to be $\{|b\rangle\}$, will always be the Fourier basis [42, 45], $|b\rangle = \sum_{a=0}^{N-1} |a\rangle \exp(i2\pi ab/N)/\sqrt{N}$. In this case, $\theta_{ab} = -2\pi ab/N$, where a and b are integers solely used to enumerate the states such that $0 \le a, b \le N-1$. With these choices for our complementary bases the density operator is simply related to the Dirac distribution by a Discrete Fourier Transform,

$$\rho_{a_1 a_2} = \sum_{b=0}^{N-1} S_\rho(a_1, b)\, e^{i2\pi b(a_1 - a_2)/N}. \tag{10}$$

This explicitly shows that average result of the joint weak

measurement, $\langle S_{ab}\rangle_\rho$, contains the same information as the density operator.

However, there is no need to ever transform to the density operator as all of the system's properties can be found directly with the Dirac distribution, $S_\rho(a,b)$. It can be used to directly calculate the average result of measuring an observable O on state ρ:

$$\langle O\rangle = N\cdot\sum_{a,b=0}^{N-1} S_\rho(a,b)\, S_O(a,b)^*, \tag{11}$$

where $S_O(a,b)$ can be calculated by Eq. (7), and * is the complex conjugate. With O chosen to be the identity operator I, we find that $\sum_{a,b=0}^{N-1} S_\rho(a,b) = Tr[\rho] = 1$. In other words, the Dirac distribution is normalized in the same manner as a probability distribution. With $O=\rho$, we find the purity $\mu = N\cdot\sum_{a,b=0}^{N-1}|S_\rho(a,b)|^2$, which reaffirms that purity is a global property of the density operator and thus, we are unable to measure purity without completely determing ρ. And finally, with $O=|a\rangle\langle a|$ or $|b\rangle\langle b|$, we see that the a and b marginals are equal to the probability distributions of outcomes a and b, e.g. $\sum_{a=0}^{N-1} S_\rho(a,b) = \text{Prob}(b)$. Consequently, the result of our joint weak measurement, the Dirac distribution of the density operator, is a capable alternative to the standard quantum quasi-probability distributions, such as the Wigner function. Its peculiarity is that it is complex whereas probabilities are real. Nonetheless, our method for directly measuring it provides an operational meaning to both its real and imaginary parts; they appear right on our measurement apparatus, in the shifts in the two conjugate observables of the pointer, e.g., x and p.

One cause for concern in our method is that S_{ab} is not Hermitian (i.e. $F^\dagger = F$, where $\dagger$ indicates the Hermitian Adjoint). According to the postulates of quantum mechanics this means it can not be an observable [46]. The typical justification for this postulate is that only a Hermitian operator has real eigenvalues. This is critical for two reasons: One, Hamiltonians containing these observables must be Hermitian in order to conserve probability. This argument has been challenged recently in regards to PT symmetry, [47]. And two, because we only ever record real values in experiments. An established counter-example to this justification is the weak value, Eq. (1), which can be complex and yet can be measured straightforwardly. One way to view the familiar complex values in quantum mechanics (e.g., in the Schrdinger equation) is as a mechanism for mathematical simplification, enabling the combination of coupled equations into one equation. In the spirit of this view, the complex weak value is a just a simplified expression for two real average measurement outcomes. Complex eigenvalues create no inconsistencies in this simplified formulation. Consequently, instead of Hermitian we contend that weakly measured observables are only required to be Normal, $FF^\dagger = F^\dagger F$.

The above discussion assumes that the weak measurement of the non-Hermitian operator S_{ab} is actually possible. It is reasonable to question whether, in principle, there is a measurement apparatus that can jointly implement a weak measurement of $|a\rangle\langle a|$ and $|b\rangle\langle b|$? The answer is yes. One possibility is to have the same measurement pointer coupled to both A and B variables [48]. Another possibility is to weakly measure a and b projectors separately, coupling each to its own measurement pointer, and then evaluate the average correlations in the two pointers (e.g., $\langle x_1 p_2\rangle$ and $\langle x_1 x_2\rangle$)[36, 49]. Both options have been demonstrated experimentally [29, 50]. Either possibility allows one to find the average result of the weak measurement of S_{ab}.

We now return to the operator ordering in the Dirac distribution, which as it stands is chosen arbitrarily. An examination of Eq. (7) reveals that in our definition of S_{ab} the measurement of A precedes the measurement of B. One might hypothesize that the order in which two weak measurements conducted would not matter since the wavefunction is nominally unchanged by both measurements. In Ref. [49], the authors considered sequential weak measurements of observables E and F and three possibilities were distinguished: measure E then F; measure F then E; and, measure E and F simultaneously. By analyzing each of these with the Von Neumann model of measurement, they found that each had a different weak value if E and F do not commute. In the context of our direct measurement of the Dirac distribution, the operator ordering choice is, thus, set by the choice of time-ordering in which one conducts the measurements of A and B.

Experimentally implementing a joint weak measurement can be challenging since the signal to noise of the weak measurement of n operators $\text{SNR}(n)$ goes as $\text{SNR}(1)^n$. We now outline an easier method to determine the density matrix. It consists of scanning the post-selected state in our original method for the direct measurement of the wavefunction. In this scenario, we replace b_0 with b in Eq. (6), where $\{|b\rangle\}$ is the Fourier basis. The weak value is given by, $P(b)\langle\pi_a\rangle_\rho^b = S_\rho(a,b)$, where the probability to post-select $B=b$ is $P(b)=\rho_{bb}$. To consider the experimental meaning of the left-hand side of this equation, we return to the general formula for the weak value, Eq. (5). In the von Neuman model, the pointer shift, which indicates the weak value of a general observable A, is determined by measuring the average of some pointer observable $D=\sum_i d_i|d_i\rangle\langle d_i|$ (e.g. $D=X$) in the subset of systems with outcome $C=c$ or $\langle A\rangle_\rho^c = k\sum_i d_i P(d_i|c)$, where k is a proportionality constant accounting for the strength of the measurement. Then using Bayes' Theorem, we find $\langle c|A\rho|c\rangle = k\sum_i d_i P(d_i,c)$, where $P(h,g)$ means the probability of getting both h and g outcomes in one trial. Applying this result to the weak measurement of $|a\rangle\langle a|$

while post-selecting $B = b$ we find,

$$S_\rho(a, b) = k \sum_i d_i P(d_i, c). \quad (12)$$

Consequently, again we can directly measure the Dirac distribution, though not through the usual evaluation of the weak value. Instead one records the joint probability for outcome c on the system of interest and outcome d_i on the pointer, when weak measuring $|a\rangle\langle a|$. In this way we can completely determine the density operator through Eq. (10). This determination has a key advantage over standard quantum state tomography in that it only requires measurements in two of the system's bases.

In summary, by measuring a pair of complementary variables of a system it is possible not only to directly measure its wavefunction but, also, to determine its density operator. To measure over the extent of wavefunction we only need to scan the first variable. To measure the density operator we must, additionally, scan the second variable. Whereas the first variable must be measured weakly, the second can either be strongly or weakly measured. The latter option corresponds to the measurement of the product of projectors for two complementary observables, such as x and p. This product operator is a non-Hermitian operator and thus has no corresponding observable relevant for standard measurement. Nonetheless, it is weakly measurable, the average result of which is a two-dimensional complex distribution that was first introduced by Dirac. With the distribution one can directly predict the outcome of any measurement on the system through a simple overlap integral, much the same as how we use the Wigner function for predictions. An advantage that this method has over other state determination techniques is that the amount of state disturbance can be minimized. Thus, in principle, we can characterize quantum states in situ, for instance, in the middle of quantum computation circuits, or during chemical reactions, without disturbing the process in which they feature.

We thank Aephraim Steinberg for useful discussions.

* jeff.lundeen@nrc-cnrc.gc.ca

[1] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature **474**, 188 (2011).

[2] J. S. Lundeen, *Generalized Measurement and Post-selection in Optical Quantum Information*, Ph.D. thesis, University of Toronto (2006).

[3] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989); D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); G. Breitenbach, S. Schiller, and J. Mlynek, Nature **387**, 471 (1997); A. G. White, D. F. V. James, P. H. Eberhard, and P. G. Kwiat, Phys. Rev. Lett. **83**, 3103 (1999).

[4] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[5] K. Husimi, Proc. Phys.-Math. Soc. Japan **22**, 264 (1940).

[6] R. J. Glauber, Phys. Rev. **131**, 2766 (1963); E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).

[7] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).

[8] K. Laiho, K. N. Cassemiro, D. Gross, and C. Silberhorn, Phys. Rev. Lett. **105**, 253603 (2010).

[9] E. Mukamel, K. Banaszek, I. A. Walmsley, and C. Dorrer, Opt. Lett. **28**, 1317 (2003); B. J. Smith, B. Killett, M. G. Raymer, I. A. Walmsley, and K. Banaszek, *ibid.* **30**, 3365 (2005).

[10] U. Leonhardt, *Measuring the quantum state of light*, Cambridge studies in modern optics (Cambridge University Press, 1997); J. F. Kanem, S. Maneshi, S. H. Myrskog, and A. M. Steinberg, J. Opt. B **7**, S705 (2005).

[11] S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, and R. Simon, J. Phys. A **39**, 1405 (2006).

[12] L. M. Johansen, Phys. Rev. A **76**, 012119 (2007).

[13] A. Walther, J. Opt. Soc. Am. **58**, 1256 (1968).

[14] E. Wolf, Phys. Rev. D **13**, 869 (1976).

[15] H. F. Hofmann, ArXiv e-prints (2011), arXiv:1107.0536 [quant-ph].

[16] A. Moyal, *Maverick mathematician: the life and science of JE Moyal* (ANU E Press, 2006).

[17] P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980); D. Walls and G. Milburn, *Quantum optics* (Springer, 2008).

[18] S. Youssef, Mod. Phys. Lett. A **9**, 2571 (1994); R. Mir, J. S. Lundeen, M. W. Mitchell, A. M. Steinberg, J. L. Garretson, and H. M. Wiseman, New J. Phys. **9**, 287 (2007).

[19] J. Wheeler and W. Żurek, *Quantum theory and measurement*, Princeton series in physics (Princeton University Press, 1983).

[20] C. A. Fuchs and A. Peres, Phys. Rev. A **53**, 2038 (1996); C. A. Fuchs and K. Jacobs, Phys. Rev. A **63**, 062305 (2001); K. Banaszek, Phys. Rev. Lett. **86**, 1366 (2001).

[21] Y. Aharonov, S. Popescu, and J. Tollaksen, Phys. Today **63**, 27 (2010); A. Cho, Science **333**, 690 (2011).

[22] D. R. Solli, C. F. McCormick, R. Y. Chiao, S. Popescu, and J. M. Hickmann, Phys. Rev. Lett. **92**, 043601 (2004).

[23] N. Brunner, A. Acin, D. Collins, N. Gisin, and V. Scarani, Phys. Rev. Lett. **91**, 180402 (2003); N. Brunner, V. Scarani, M. Wegmuller, M. Legre, and N. Gisin, *ibid.* **93**, 203902 (2004).

[24] H. M. Wiseman, Phys. Rev. A **65**, 032111 (2002).

[25] A. M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995); Phys. Rev. A **52**, 32 (1995).

[26] H. M. Wiseman, Phys. Lett. A **311**, 285 (2003).

[27] N. S. Williams and A. N. Jordan, Phys. Rev. Lett. **100**, 026804 (2008); A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nature Phys. **6**, 442 (2010); M. E. Goggin, M. P. Almeida, M. Barbieri, B. P. Lanyon, J. L. O'Brien, A. G. White, and G. J. Pryde, Proc. Natl. Acad. Sci. U.S.A. **108**, 1256 (2011).

[28] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Phys. Lett. A **324**, 125 (2004).

[29] J. S. Lundeen and A. M. Steinberg, Phys. Rev. Lett. **102**, 020404 (2009).

[30] O. Hosten and P. Kwiat, Science **319**, 787 (2008); P. B. Dixon, D. J. Starling, A. N. Jordan, and

J. C. Howell, Phys. Rev. Lett. **102**, 173601 (2009); A. Feizpour, X. Xing, and A. M. Steinberg, Phys. Rev. Lett. **107**, 133603 (2011).
[31] G. A. Smith, S. Chaudhury, A. Silberfarb, I. H. Deutsch, and P. S. Jessen, Phys. Rev. Lett. **93**, 163602 (2004); G. G. Gillett, R. B. Dalton, B. P. Lanyon, M. P. Almeida, M. Barbieri, G. J. Pryde, J. L. O'Brien, K. J. Resch, S. D. Bartlett, and A. G. White, Phys. Rev. Lett. **104**, 080503 (2010).
[32] N. W. M. Ritchie, J. G. Story, and R. G. Hulet, Phys. Rev. Lett. **66**, 1107 (1991).
[33] G. J. Pryde, J. L. O'Brien, A. G. White, T. C. Ralph, and H. M. Wiseman, Phys. Rev. Lett. **94**, 220405 (2005).
[34] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[35] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).
[36] J. S. Lundeen and K. J. Resch, Phys. Lett. A **334**, 337 (2005).
[37] R. Jozsa, Phys. Rev. A **76**, 044103 (2007).
[38] O. Oreshkov and T. A. Brun, Phys. Rev. Lett. **95**, 110409 (2005).
[39] H. F. Hofmann, Phys. Rev. A **81**, 012103 (2010).
[40] J. Dressel, S. Agarwal, and A. N. Jordan, Phys. Rev. Lett. **104**, 240401 (2010).
[41] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, 1955).
[42] T. Durt, B. Englert, I. Bengtsson, and K. Zyczkowski, Int. J. Quantum Inf. **8**, 535 (2010).
[43] A. Romito and Y. Gefen, Physica E **42**, 343 (2010), proceedings of the international conference Frontiers of Quantum and Mesoscopic Thermodynamics FQMT '08; E. Haapasalo, P. Lahti, and J. Schultz, Phys. Rev. A **84**, 052107 (2011); S. Wu and Y. Li, **83**, 052106 (2011).
[44] U. Leonhardt, Phys. Rev. Lett. **74**, 4101 (1995).
[45] I. Bengtsson, in *Foundations of Probability and Physics - 4*, American Institute of Physics Conference Series, Vol. 889, edited by G. Adenier, C. Fuchs, & A. Y. Khrennikov (2007) pp. 40–51.
[46] R. Shankar, *Principles of quantum mechanics* (Plenum Press, 1994).
[47] C. Bender, D. Brody, and H. Jones, Am. J. Phys. **71**, 1095 (2003); C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature. Phys. **6**, 192 (2010).
[48] K. Molmer, Phys. Lett. A **292**, 151 (2001); Y. Aharonov, A. Botero, S. Popescu, B. Reznik, and J. Tollaksen, **301**, 130 (2002).
[49] K. J. Resch and A. M. Steinberg, Phys. Rev. Lett. **92**, 130402 (2004); G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A **76**, 062105 (2007); G. Mitchison, Phys. Rev. A **77**, 052102 (2008).
[50] K. Yokota, T. Yamamoto, M. Koashi, and N. Imoto, New J. Phys. **11**, 033011 (2009).